In *Q is for Quantum* I chose to present a version of quantum nonlocality (Bell's theorem) using Hardy's paradox, instead of the version we typically do in experiment, which is called the CHSH inequality. This is because Hardy's paradox can be presented as a "logical impossibility", while the CHSH inequality is basically a "statistical violation of common sense". Most people, including me, suck at statistics, and so I worried that this amazing feature of the world might be discarded by the reader as either too hard, or somehow "only statistical". I had started off doing the CHSH version, and this document contains most of that initial rough draft. However, the calculation at the end requires the "squaring rule" introduced in Part II of the book. So this document is not as clear and self-contained as I would like, but it will potentially help any teachers who wanted to use this example.
- *Terry Rudolph, Feb 2021*

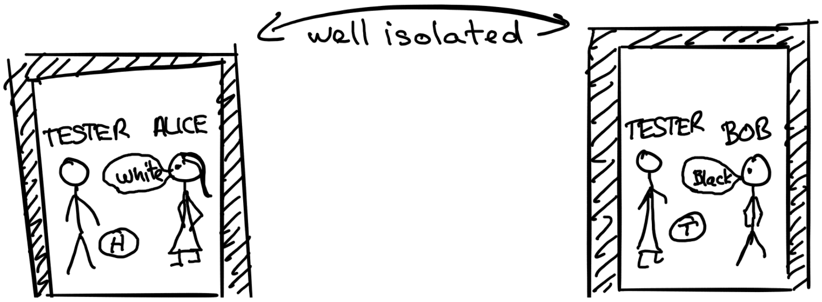
Q-Nonlocality (Bell's Theorem or the CHSH inequality) using the misty states formalism

A tale of telepathy

Many people - lets call them psychics - claim telepathic ability, that is, an ability for instantaneous communication between two separated minds, via means unknown. Whenever asked to evaluate claims of powers that go beyond the laws of physics as we currently understand them, the prominent skeptic and magician James Randi (who offers a $1 million dollar prize for any demonstration of such) will insist the claimants first make precise exactly what it is they say they can do. He has found there is no point in a skeptic designing the test and then challenging the psychics to meet its standards, because the psychics can wriggle out by saying their powers are not compatible with meeting the specific challenge the skeptic would like. Rather, it is best to test exactly what it is the psychics claim to be able to do, making sure only to install obvious and agreed-upon safeguards against cheating.

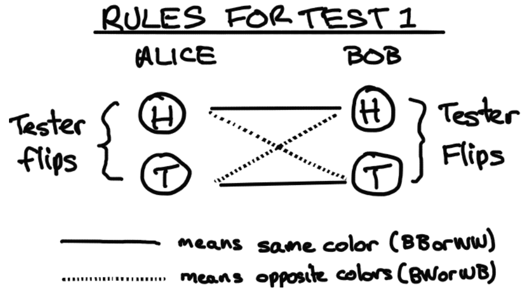
Imagine (as I hope is actually true) that you are skeptical of generic claims of psychic abilities, but you are open to being convinced otherwise by a suitably rigorous demonstration. You are working for Randi when two psychics, Alice and Bob, contact you (by regular means), claiming to be telepathic. You now enter a negotiation as to how they will demonstrate their ability. Unfortunately, and this is quite typical, they do not claim to be able to do something obvious and readily testable, like transmit a message about whether they have been shown a picture of a dead cat versus an alive cat. Their powers, they say, are subtler.

Eventually the protocol they propose involves them each being separated in well-shielded rooms, to prevent communication by any regular means. Within each room a tester will flip a coin and tell the psychic in that room the outcome, "heads" or "tails". The psychics will each then have to then say to their tester one of two very magical words, namely either "black" or "white".



So far so good. They then propose they will have demonstrated they are psychic when their answers obey the following conditions:

Proposed test of telepathy 1: *When the two coin flip outcomes are the same, then the psychics will always announce the same color, when the two coin flip outcomes are not the same, they will always announce different colors.*

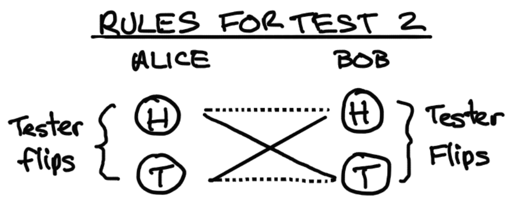


If Alice and Bob are telepathic they can certainly win this "game" every time - Alice just telepathically tells Bob whether her tester flipped heads or tails, and which color she said. In fact she need only tell him the coin flip outcome, they can pre-decide the colors to assign to each flip. So would you agree that this was a suitable test of telepathy? Hopefully not. The issue is that no telepathic communication is required for the psychics to win the game every time it is played. All they need to do is pre-agree that if the coin flip in their room is heads they will announce "white" and if it is tails they will announce "black".

After you point this out to them, Alice and Bob come back with a slightly different proposal:

Proposed test of telepathy 2: *When the two coin flip outcomes are the same, then the psychics will always announce different colors, when the two coin flip outcomes are not the same, they will always announce the same color.*

Would you now agree that passing this test demonstrates telepathy? Again, hopefully not, because once again there is a simple way for them to win the game every single time that requires no telepathic communication. They need only agree that Alice will say "black" and Bob will say "white" when each psychic's respective tester flips heads, while Alice will say "white" while Bob will say "black" when their respective tester flips tails.



How to win test 2 without telepathy:

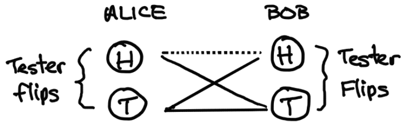
Tester	Alice	Tester	Bob
Heads	Black	Heads	White
Tails	White	Tails	Black

After once again pointing out this is not a suitable test of telepathy, Alice and Bob come back with:

Proposed test of telepathy 3: *When the coin flip outcomes are both heads, then the psychics will always announce different colors; when either (or both) of the coin flip outcomes are tails they will always announce the same color.*

Now the rules for this game are a bit more complicated, and I would encourage you to stop now and think about possible strategies for Alice and Bob. You should try and prove that they cannot pass the test (win the game, as it were) for certain.

RULES FOR TEST 3



TRY to win test 2 without telepathy:

Tester	Alice	Tester	Bob
Heads	Black	Heads	White
Tails	????	Tails	Black

There are a couple of different ways to analyze this third game (assuming Alice and Bob are not actually telepathic!). One such analysis goes: Imagine Bob will say “white” if his tester says “tails”. Bob’s answer when he is told “tails” must agree with Alice’s answer, regardless of which flip outcome she gets, so she must answer “white” for both “heads” and “tails”. But then what color can Bob answer for “heads”? If he answers “white” it will violate the rule they must give different colors when both get told “heads”, and if he answers “black” it will violate the rule they must give the same color when Alice gets “tails”. Changing Bob’s initial color choice for “tails” to “black” won’t help, it will just interchange black and white in the whole argument. In the figure I have indicated a different attempt, that also doesn’t work.

I recommend grabbing a piece of paper and trying yourself to find a strategy for Alice and Bob where they can win every time. Because, while you may think this is all a bit silly, it will take us somewhere really amazing. You need to completely convinced either by my argument, or your own analysis, that the psychics cannot win the game every time, and in fact at most they could expect to win 3 out of 4 games on average, or three-quarters of the time. In fact they can win the game three-quarters (75%) of the time by just answering “white” all the time (meaning they only lose when both testers flip heads). For this reason we will need to test them playing the game many times - if we only test them a very small number of times (or even just once), there is some chance the psychics do win every time.

Let me harp on a little longer about the psychics’ options if they are not telepathic. Perhaps it is a mistake for them to pre-determine their strategy? Maybe they should only decide on a “black” versus “white” answer once they know the coin flip they are told by their tester? For example, they could use a coin flip of their own and base their color choice partially on the outcome. Can you see why such a “non-deterministic” (i.e not pre-determined) strategy won’t really help? In effect it makes the other psychic even less sure what their partner is answering, and that cannot help them win in the long run. It could, if they got very lucky, help them win in the short run. Once again we see we need to test them multiple times to be sure they didn’t just get lucky.

Understanding all this, you and Randi agree that if they can win this third game one hundred times in a row then they will win the $1 million.

At this point, the psychics come back and say something to the effect of “Well we aren’t sure we can win it every single time, but we can use our telepathic powers to win more than 80% of the time”.

Immediately you get suspicious, because you know that just by getting lucky the psychics might hope to win 80 or more out of 100 trials, even though on average we would only expect them to win 75 (three-quarters of 100) times. Winning an extra 5 times just by pure luck doesn't seem so unlikely.

But hopefully if you increase the number of trials you should eventually be able to decide for sure whether they are winning 75% of the time, or whether they are winning more like 80% of the time. For instance, if you make them play 1000 times, then if they are not psychic you and Randi expect them to win around 750 times, while they claim they can win more than 800 times. Should you give them the money if they do? Maybe they could get lucky and win the extra 50 times? An extra 50 wins is definitely less likely than an extra 5 wins, but perhaps it can still be achieved if they are lucky? If so, exactly how lucky would they need to be?

Calculating odds like this is tricky. Fortunately, you have your friend in the bank, the same one who gave you inside information when you were robbing them in Part I. Unlike most people who work in banking, your friend actually understands this kind of thing. He suggests that you have the psychics play the game 10000 times. Then without telepathy they would win on average 7500 times. He calculates the chance of them exceeding 8000 wins, and finds it is absolutely and utterly, mind-numbingly ridiculously small. It is so small that you have more chance winning a game where I take one grain of sand, mark it somehow, and hide it anywhere on any beach in the whole world, or anywhere in the Sahara desert as well. You then walk around the whole world blindfolded, sifting through all those sandy beaches, dragging yourself through that lovely desert, and at some point you grab a single grain of sand. The chance that you grab the same grain that I marked is still greater than the chance the psychics can win the game more than 8000 in 10000 times.

Once you understand all this you and Randi agree with the psychics that this is a fair test. You even get a bit carried away, and offer to throw in a couple of genuine gold bars you came across recently as well.

Playing the games

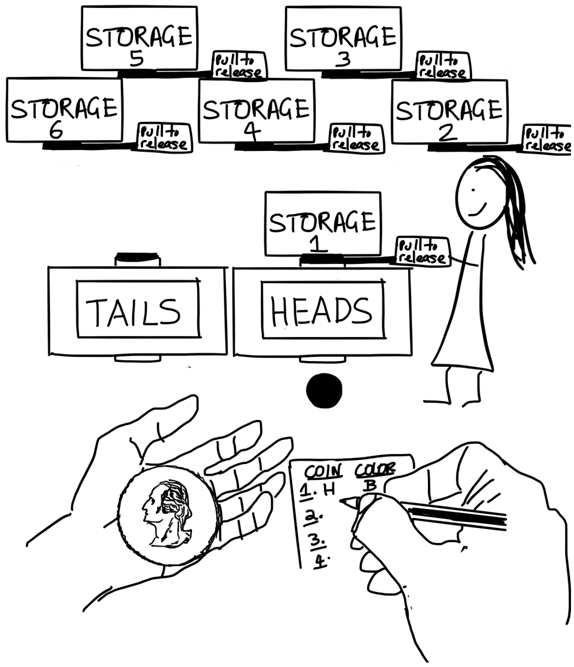
The day of the test comes, and there is much fanfare as the world's media descends. Alice and Bob show up. Hang on, what is this? They are each carrying a very large number of boxes that they want to take into the room with them. They say that there is nothing in the rules that does not let them have some `telepathic aids'.

Now, given more time and without the glare of the media, you would hopefully realize that you should contact me or some other nerd just to be sure you haven't missed something. But here is the thing: You and Randi are fully convinced that you have managed to completely isolate the two rooms that each psychic/tester pair will be closed into. What could it really matter if Alice and Bob bring some stuff in with them? Even if they bring in powerful supercomputers to help them do complicated calculations that might somehow help determine the color they announce to each coin flip, you are sure from the arguments presented above that they cannot win each individual game more than 75% of the time. Of course you understand from your bank escapade that a misty computer can do some calculations faster than a supercomputer, but all the computing power in the world isn't going to change the fact they ultimately need to answer "black" or "white" according to some simple rules that are easily shown to not be consistently achievable. So you let the test go ahead.

Since you and Randi are the ones with serious money at stake, you have decided as an extra hedge against possible cheating ("hey skeptic tester friend, want half a million bucks and a gold bar?") that you each will act as a tester.

When you get into the room with Alice you find that she has brought in a huge pile of small boxes, each labeled STORAGE and numbered from 1 to 10000. There are also two larger boxes, one labeled HEADS and the other TAILS, each of which have a hole in the top and a hole in the bottom.

The test begins. You flip your coin and it comes up “heads”, which you call out to Alice. Alice then takes the small box labeled STORAGE 1 over to the HEADS box. She places it right above the top hole and then pulls some kind of lever on the side of the STORAGE box. Almost immediately a black ball falls out from the bottom of the HEADS box. “My first answer is black” Alice tells you. You write it on the piece of paper you brought to record the coin flips and corresponding black/white answers.



You can only assume that the ball which fell out from the bottom of the TAILS box came from within the STORAGE 1 box. But you didn’t manage to get a glimpse of it before it fell through. The next time you try a bit harder, but to no avail. This time the coin comes up tails and Alice takes STORAGE 2, holds it just above the TAILS box and pulls the lever. This time the ball that drops out the bottom is white and Alice says “white” is her answer. For someone supposedly being telepathic Alice is acting quite brusque and business-like. “Next” she says impatiently.

After a full day, with the test drawing to a close, you have seen the same procedure repeated thousands and thousands of times. Each time, according to the coin flip you announce, Alice holds the next unused storage box above the HEADS or TAILS box, pulls some kind of release lever and waits for a ball to drop out the bottom. She announces “black” or “white” according to the color of that ball.

After the end of the 10000 repetitions of the game you and Randi meet up and exchange stories. He had the exact same experience with Bob as you had with Alice - Bob also based his black/white announcements on the color of a ball which fell out of a storage box held above a HEADS or TAILS box.

You and Randi now sit down and start the laborious process of counting up how many times the psychics won. That is, how many times did they say the same color when one or the other of you flipped tails, but opposite colors when both of you flipped heads?

When you add it all up you find that the psychics have won 8120 times, which far exceeds the 8000 times you agreed they needed to exceed to win the cash and the gold bars. A little nervous (you, not Randi, he's chilled about everything) you do the adding up again - maybe having that drink while doing math wasn't a great idea and you just messed up the counting? (For that matter, are you old enough to drink a beer yet?) You find that you were correct the first time, the psychics have definitely exceeded 7500 wins by a long way.

I don't know about you, but if it was me I would be very, very suspicious at this point. Randi you can likely trust, although maybe given what is at stake even he should be suspect. Because remember, what is at stake is not just money, it is something far more important: it's the possibility of some kind of psychical connection that defies the known laws of physics.

What went wrong?

Assuming for the moment you trust Randi, your suspicions will first fall on the isolation rooms. All it takes to win the game every single time is for information about the coin flip outcome in the other room to be available. For instance, perhaps hidden inside the HEADS and TAILS boxes is a cellphone of some form, which sends a message to the other room? While you have completely shielded the rooms to all known types of signals, there could be ones you don't know.

On the face of it there will never be a way to completely shield a room. However, all known signals capable of carrying information share one common feature - they travel at a speed no faster than the speed of light. Now light travels very fast, but not infinitely fast. So one way to ensure no signal is sent is the following. You demand that after you tell Alice the coin flip outcome, she tells you her black/white answer (in effect the ball color) before there is time for a signal travelling at the speed of light to make it to the room where Bob and Randi are.

Because light travels so fast, even if you put the isolation rooms at the opposite ends of the earth there are only small fractions of a second during which the whole process must take place. I doubt you can even flip a coin fast enough. For this reason, when we do this experiment we use some electronic equipment to do the coin flip. We also use a mechanical version of Alice who can move very fast and who doesn't get tired. But in principle what we do is identical to the procedure I have just descried, and if we could afford to send Bob and Randi to Mars, then we would have plenty of time - up to half an hour for each iteration of the game, because Mars is so far away it takes even light from there a long time to get to us, and vice versa. When we do this version of a perfectly isolated experiment, we still find that Alice and Bob are winning the game waaaaay more than the 75% of the time that should be the maximum if there is no telepathic link between the rooms.

So how is it being done? The answer, as you may have guessed already, involves misty states of balls somehow.

Let's get entangled up in the question: what is inside the storage boxes?

Inside each storage box is a single ball. The balls within the storage boxes of the same number (remember Alice and Bob each have a storage box numbered 1, 2,… 10000) have been carefully prepared in the misty state:

`[WW,WB,WB,BW,BW,-BB]`

The storage boxes are very carefully designed to not inadvertently look at the ball colors. As we know, looking at the color will destroy the mist. That is, if the storage boxes destroy the mist, then it would be as if they had been prepared randomly in one of the configurations in the mist. This would equate to a non-misty strategy which our psychics could have employed, and which we already saw could not help Alice and

Bob pass the 75% winning threshold. Somehow the psychics need to keep the mist "alive" until they know what the coin flip is, and the storage boxes do that.

After preparing 10000 pairs of balls in this way, Alice and Bob are "sharing" a giant misty state. We could combine all of this into a single mist, using the rules given in Part I, but it would be huge, and in fact this is unnecessary because Alice and Bob never do anything that involves bringing together balls from two different such mists. They simply drop each ball through one of the two boxes labelled HEADS or TAILS. This means (though I am aware it is not at all obvious) that we can just focus on what happens to a single such pair of balls.

Now that we know what the storage boxes contain – they each contain a single ball which is prepared in a special misty state with a ball held in a matching numbered storage box held by the other psychic - we need to know what happens to each ball when they pass through the HEADS and TAILS boxes.

What goes on inside the HEADS & TAILS boxes?

In fact the HEADS box does nothing – the ball just drops through it and the color is observed, while the TAILS box is, you guessed it – a PETE box.

In Part II of *Q is for Quantum* I explained the somewhat strange "squaring rule" for working out the probabilities of seeing a particular configuration when you observe a mist that does not have all configurations appearing the same number of times. That is the case for the mist [WW,WB,WB,BW,BW,-BB]. WB and BW each appear twice, while WW and BB appear once. It would be good practice to review that rule and then see if you can compute the various probabilities I am about to tell you on your own.

If both psychics are told Heads, then they are simply observing the balls in their mist. Remember they win if they give opposite color answers – so they win if they observe BW or WB. The probability of them seeing WB is 4/10, while of them seeing BW it is 4/10. Therefore the probability they win when you and Randi both flip heads is 8/10 or 80%.

What if you flip tails and Randi flips heads. Alice's TAILS box is just a PETE box. So we can compute how the mist evolves, and we find it becomes

`[WW,WW,WW,WB,-BW,BB,BB,BB]`

This time the psychics win if they give the same answer, and you can compute the probability of them answering WW or BB to be 9/10 or 90%. They have the same chance of winning if the situation is reversed – you flip heads and Alice flips tails because in that case the mist evolves to

`[WW,WW,WW,-WB,BW,BB,BB,BB]`

Finally what happens if you and Randi both flip tails? Now both psychics are putting their ball through a PETE box, and so the calculation is messy. But if you do it right it becomes

`[WW,WW,WW,WW,WB,WB,BW,BW,-BB,-BB,-BB,-BB]`

From this you can deduce that the probability of them winning is 8/10 or 80%.

We see then that regardless of what coin flip outcomes you get they have at least 80% chance of winning, possibly greater. And this exceeds the 75% bound which we deduced previously was "obviously" the best they could do.

Just saying "misty states let them do it" doesn't really explain much. At this point you may want to revisit the discussion in the latter half of Part II.